The Number of Small Covers over Cubes

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Backgrounds

Small cover : a closed smooth manifold M of dim. n with a locally standard smooth $(\mathbb{Z}_2)^n$-action whose orbit space $M/(\mathbb{Z}_2)^n$ is a simple polytope.

- P : simple polytope of dim n.
- $\mathfrak{F}(P) = \{F_1, \dots, F_m\}$: the set of facets of P.
- $T = \mathbb{Z}_2^n$: real torus of dim. n.
- $\lambda : \mathfrak{F}(P) \to H_2(BT) = \mathrm{Hom}(\mathbb{Z}_2, T) = \mathbb{Z}_2^n$: Characteristic Function if $\cap F_i$ is a vertex $\Rightarrow \{\lambda(F_i)\}$ is a basis of $\mathbb{Z}_2^n$

Let $cf(P)$ denote the set of characteristic functions over P.

M.Davis and T.Januszkiewicz (1991)

All small covers over P are given by $\{M(\lambda) | \lambda \in cf(P)\}$.

- M_1, M_2 : small covers over P.

M_1 and M_2 are equivalent up to ... :

1. weakly equivariant homeo. (or weak $\mathbb{Z}_2^n$ - homeo.) if
$$\exists f : M_1 \to M_2 \text{ s.t. } f(t \cdot x) = \varphi(t) \cdot f(x),$$
where $\varphi \in \mathrm{Aut}(\mathbb{Z}_2^n)$ (In particular, equivariant homeo. if $\varphi = Id$).
2. D-J equivalence if $\exists f$: weakly equivariant homeo. s.t.
$$M_1 \xrightarrow{f} M_2 \text{ over } P$$

The set $cf(P)$ has two group actions, related to two different sets of equivalence classes of small covers over P.

- $GL(n, \mathbb{Z}_2)$: the general linear group on $\mathbb{Z}_2^n$ - Left action by $\mathbb{Z}_2^n$-basis changing .
- $Aut(\mathfrak{F}(P))$: the group of automorphisms of $\mathfrak{F}(P)$ - Right action by composition.

Well-known Facts

There are 1-1 correspondences between
$$\{\text{D-J classes over } P\} \longleftrightarrow GL(n, \mathbb{Z}_2) \backslash cf(P)$$
$$\{\mathbb{Z}_2^n\text{-homeo. classes over } P\} \longleftrightarrow cf(P)/Aut(\mathfrak{F}(P)).$$

One may assign an $(n \times m)$-matrix Λ to an element $\lambda \in cf(P)$ by ordering the facets and choosing a basis for $\mathbb{Z}_2^n$.
$$\Lambda = (\lambda(F_1) \cdots \lambda(F_n)) = (A|B),$$
where A be a $(n \times n)$-matrix and B be a $(n \times (m-n))$-matrix. Note that $\Lambda \sim (E_n | A^{-1}B)$ up to D-J equivalence. Denote $\Lambda_* = A^{-1}B$.

When $P = I^n$, Λ_* is $(n \times n)$ matrix. By labeling with $F_i \cap F_{n+i} = \phi$ for all i, we have the following relation:

The non-singularity condition $\Leftrightarrow$ Every principal minor of Λ_* is 1.

The **main purpose** is to count

1. small covers over I^n and $\prod_{i=1}^m \Delta^{n_i}$,
2. orientable small covers over I^n, and
3. equivalence class over I^n up to $\mathbb{Z}_2^n$-homeo.

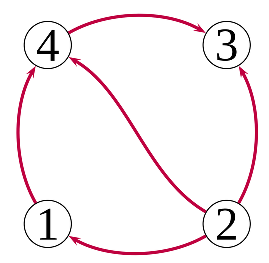
The number of small covers

Digraph : Graph with at most one edge directed from vertex i to vertex j, for $1 \le i \le n, 1 \le j \le n$.
Acyclic : Graph without cycles of any length.

- $m(n)$: set of matrices all of whose principal minor is 1.
- $\mathfrak{G}_n$: the set of acyclic digraphs with n labeled nodes.
- $A(G)$: The vertex adjacency matrix of $G \in \mathfrak{G}_n$.

Theorem

There is a bijection $\phi : \mathfrak{G}_n \to m(n)$ by
$$\phi : G \mapsto A(G) + E_n$$
where E_n is an identity matrix of size n.



$$\xrightarrow{\phi} \begin{pmatrix} 1 & 0 & 0 & 1 \\ 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix}$$

Note that
$$\{\text{D-J classes over } I^n\} \leftrightarrow m(n) \leftrightarrow \mathfrak{G}_n$$

R.Robinson(1970) and R.Stanley(1973)

Let R_n be the number of acyclic digraphs with labeled n nodes.
$$R_n = \sum_{k=1}^{n} (-1)^{k+1} \binom{n}{k} 2^{k(n-k)} R_{n-k}$$

n	0	1	2	3	4	5	6	7	$\cdots$
R_n	1	1	3	25	543	29281	3781503	1138779265	$\cdots$

The number of small covers over a product of simplices

$$\sharp DJ\Big(\prod_{i=1}^{\ell} \Delta^{n_i}\Big) = \sum_{G \in \mathfrak{G}_\ell} \prod_{v_i \in G} (2^{n_i} - 1)^{\mathrm{outdeg}(v_i)}$$

$\underline{m=2}$: $\sharp DJ(\Delta^{n_1} \times \Delta^{n_2}) = 1 + (2^{n_1} - 1) + (2^{n_2} - 1)$.
$\underline{m=3}$: $\sharp DJ(\Delta^{n_1} \times \Delta^{n_2} \times \Delta^{n_3}) = 1 + 2(x_1+x_2+x_3) + (x_1+x_2+x_3)^2 + (x_1x_2 + x_2x_3 + x_3x_1) + (x_1+x_2+x_3)(x_1^2+x_2^2+x_3^2) - x_1^3 - x_2^3 - x_3^3$, where $x_i = 2^{n_i} - 1$ for $i = 1, 2, 3$.

Orientable small covers

H.Nakayama and Y.Nishimura (2005)

$$M(\lambda) \text{ is orientable } \iff \exists \text{ a basis of } \mathbb{Z}_2^n \text{ s.t. } Im(\varepsilon\lambda) = \{1\}$$
where $\varepsilon : \mathbb{Z}_2^n \to \mathbb{Z}_2$ is defined by $\varepsilon(e_i) = 1$ for all i.

The number of orientable small covers over a cube

Let O_n be the number of orientable small covers over I^n.
$$O_n = \sum_{k=1}^{n} (-1)^{k+1} \binom{n}{k} 2^{(k-1)(n-k)} R_{n-k}$$

n	0	1	2	3	4	5	6	7	$\cdots$
O_n	1	1	1	4	43	1156	74581	11226874	$\cdots$

Note that an orientable small cover is correspondent to an acyclic digraph with labeled n nodes all of whose in-degree is even.

Analysis

$$\frac{O_n}{R_n} \to 0 \text{ as } n \to \infty.$$

- $F(x) = \sum \frac{x^n}{n! 2^{\binom{n}{2}}}$, $O(x) = \sum O_n \frac{x^n}{n! 2^{\binom{n}{2}}}$: chromatic generating functions.
$$O(x) F\Big(-\frac{x}{2}\Big) = 1 - F(-x)$$

Equivariant homeomorphism classes

All elements of $Aut(\mathfrak{F}(I^n))$ can be written in a simple form as follows :
$$\mu \cdot \chi_1^{\varepsilon_1} \cdot \dots \cdot \chi_n^{\varepsilon_n}, \varepsilon_j \in \mathbb{Z}_2$$
with $\mu \in S_n$ and $\chi_1^2 = \dots = \chi_n^2 = 1$. Using by Burnside's formula, we have

The number of $\mathbb{Z}_2^n$-homeomorphism classes over a cube

Let Q_n be the number of equivariant homeomorphism classes of small covers over I^n.
$$Q_n = \frac{\sum_{k=0}^{n} \binom{n}{k} 2^{k(n-k)} R_k}{2^n n!} \cdot \prod_{i=0}^{n-1} (2^n - 2^i)$$

n	0	1	2	3	4	5	$\cdots$
Q_n	1	1	6	259	87360	236240088	$\cdots$

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