Toric Cohomological Rigidity of Simple polytopes

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Cohomologically rigid polytopes

- P : simple polytope of dim n. ($\Leftrightarrow$ each vertex is the intersection of exactly n facets)
- the standard $T^n := (S^1)^n$-action on $\mathbb{C}^n$ is given by the formula

$$(t_1,\dots,t_n)\cdot(z_1,\dots,z_n) = (t_1z_1,\dots,t_nz_n)$$

for $(t_1,\dots,t_n)\in T^n$ and $(z_1,\dots,z_n)\in\mathbb{C}^n$.

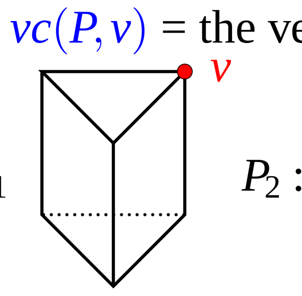
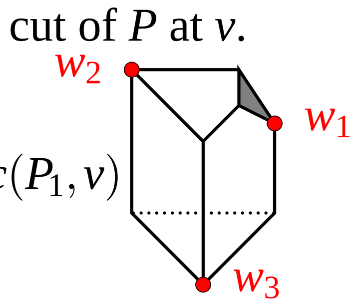
Quasitoric manifolds : a closed smooth manifold M of dim. $2n$ with a locally standard smooth T^n-action whose orbit space M/T is a simple polytope P.

- $M\to P, N\to Q$: quasitoric manifolds.

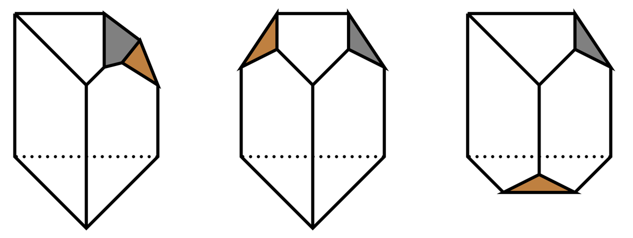
$$H^*(M)\cong H^*(N) \text{ as rings } \Rightarrow P\approx Q?$$

In general, the answer is NO! For instance,

- $vc(P,v)$ = the vertex cut of P at v.

P_1  $P_2 := vc(P_1,v)$ 

$M_1 = \mathbb{C}P^2\times\mathbb{C}P^1 \xrightarrow{\pi} P_1$ $M_2 =$ blow up of M_1 at $\pi^{-1}(v)$ $\xrightarrow{\pi'} P_2$

$P_{3,i} := vc(P_2,w_i)$ 

$M_{3,i} =$ blow up of M_2 at $\pi'^{-1}(w_i) \to P_{3,i}$

$M_{3,i}\cong(\mathbb{C}P^2\times\mathbb{C}P^1)\sharp\mathbb{C}P^3\sharp\mathbb{C}P^3$

Def. (Cohomological) Rigidity of Simple polytopes

A simple convex polytope P is (cohomologically) rigid if there exists a quasitoric manifold over over P, and whenever there exists a quasitoric manifold N over another polytope Q with a graded ring isomorphism $H^*(M)\cong H^*(N)$ there is a combinatorial equivalence $P\approx Q$.

Which polytopes are rigid?

M.Masuda and T.Panov(2007)

I^n is rigid.

Rigid Polytopes

The following polytopes are rigid.

- Every polygon, i.e., 2-dim polytope.
- Every triangle-free n-dim simple polytope with facet numbers $\le 2n+2$.
- Any product of simplices $\prod_{i=1}^k\Delta^{n_i}$
- vertex cut of a product of simplices $vc(\prod_{i=1}^k\Delta^{n_i})$
- Dodecahedron

Information from cohomology rings

(Cohomological) rigidity question for quasitoric manifolds

- M,N : quasitoric manifolds

$$H^*(M)\cong H^*(N) \text{ as rings } \Rightarrow M\cong N \text{ up to homeo. (or diffeo) ?}$$

[S.Choi, M.Masuda and D.Suh (2008)]

- M : quasitoric manifold over $\prod\Delta^{n_i}$

$$H^*(M)\cong H^*(\prod\mathbb{C}P^{n_i})\Rightarrow M\cong\prod\mathbb{C}P^{n_i}.$$

Since $\prod\mathbb{C}P^{n_i}$ is rigid, we may drop the condition over $\prod\Delta^{n_i}$.

M.Davis and T.Januszkiewicz (1991)

- $H_T^*(M)=\mathbb{Z}[v_1,\dots,v_m]/I_P=\mathbb{Z}(P)$: Stanley-Reisner ring where $F_i\leftrightarrow v_i\in H_T^2(M)$ and $I=<v_{i_1}\cdots v_{i_k}|F_{i_1}\cap\cdots\cap F_{i_k}=\phi>$
- $H^*(M)=\mathbb{Z}[v_1,\dots,v_m]/I_P+J=\mathbb{Z}(P)/J$ where J is an ideal gen. by some linear combinations of $v_1,\dots,v_m$.

- f_i : the number of codim $i+1$ faces of P.

$$\sum_{i=0}^n h_it^{n-i}=\sum_{j=0}^n f_{j-1}(t-1)^{n-j}$$

M.Davis and T.Januszkiewicz (1991)

$$\dim_{\mathbb{Z}}H^{2i}(M)=h_i(P)$$

Cohomology ring determines *face numbers*!

- $A=\mathbb{Q}[x_1,\dots,x_m]$: the polynomial graded ring with $\deg x_i=2$ for all i.

A free resolution $[R,d]$ of a finitely generated A-module M is an exact sequence

$$0\to R^{-n}\xrightarrow{d}R^{-n+1}\xrightarrow{d}\cdots\xrightarrow{d}R^0\xrightarrow{d}M\to 0,$$

- N :finitely generated A-module

$$0\to R^{-n}\otimes_A N\xrightarrow{d\otimes 1}R^{-n+1}\otimes_A N\xrightarrow{d\otimes 1}\cdots\xrightarrow{d\otimes 1}R^0\otimes_A N\to 0$$

Take the cohomolgy of the sequence

$$\mathrm{Tor}_A^{-i}(M,N):=H^i(R^{-*}\otimes_A N)=\mathrm{Tor}_A^{-i,2j}(M,N).$$

The bigraded Betti numbers of P are defined by

$$\beta^{-i,2j}(P)=\dim_{\mathbb{Q}}\mathrm{Tor}_A^{-i,2j}(\mathbb{Q}(P),\mathbb{Q}).$$

Proposition

- P,P' : n-dim. simple convex polytopes
- J,J' : ideal of $\mathbb{Q}(P)$ (resp. $\mathbb{Q}(P')$) generated by a linear regular seq.

If $\mathbb{Q}(P)/J\cong\mathbb{Q}(P')/J'$ as A-algebra, then

$$\mathrm{Tor}_A^{-i,2j}(\mathbb{Q}(P),\mathbb{Q})=\mathrm{Tor}_A^{-i,2j}(\mathbb{Q}(P'),Q)$$

- $M\to P, N\to Q$: quasitoric manifolds

$$H^*(M;\mathbb{Q})\cong H^*(N;\mathbb{Q}) \text{ as rings } \Rightarrow \beta^{-i,2j}(P)=\beta^{-i,2j}(Q)$$

Cohomology ring determines *bigraded Betti numbers* of P!

Sketch of Proofs

Here, we shall show that **any finite product of simplices is rigid.** It is easy to show that any product of simplices supports quasitoric manifolds.

- $\sigma(P)=\sum_{j\ge 2}j\beta^{-1,2j}$

$2\sigma(P)$ is nothing but the sum of the degrees of all elements of a minimal basis of I_P.

Lemma

- P : simple polytope with m facets

$$\sigma(P)=m\Longleftrightarrow P=\prod\Delta^{n_i}$$

Let M be $2n$-dim. quasitoric manifold over $P=\prod_{i=1}^t\Delta^{n_i}$ and N be another $2n$-dim. quasitoric manifold over Q s.t. $H^*(M)\cong H^*(N)$. Then,

$$f_i(M)=f_i(N).$$

In particular, $\sigma(\mathbb{Q}(P))=f_0(P)=f_0(Q)=n+t$. Since $H^*(M;\mathbb{Q})=H^*(N;\mathbb{Q})$,

$$\beta^{-i,2j}(P)=\beta^{-i,2j}(Q).$$

Thus, $\sigma(\mathbb{Q}(P))=\sigma(\mathbb{Q}(Q))=f_0(Q)$. Hence $Q\approx\prod_{j=1}^s\Delta^{m_j}$. But $\beta^{-i,2j}(P)=\beta^{-i,2j}(Q)\Rightarrow\{n_i\}=\{m_j\}$ and $t=s$. Therefore $P\approx Q$.

□

Some related questions

Rigidity of Cohen-Macaulay complex

- simplicial complex K^{n-1} is Cohen-Macaulay if $\exists$ length n regular seq. in $\mathbb{Q}(K)$.

Rigidity of Cohen-Macaulay complex

K is rigid if for any L and for ideals $J_K\subset\mathbb{Q}(K), J_L\subset\mathbb{Q}(L)$ gen. by a linear regular seq. of length n,

$$\begin{array}{ccc}\mathbb{Q}(K) & \overset{\exists}{\longrightarrow} & \mathbb{Q}(L)\\ \downarrow & & \downarrow\\ \mathbb{Q}(K)/J_K & \overset{\cong}{\longrightarrow} & \mathbb{Q}(L)/J_L\end{array}$$

Question asked by Buchstaber

[V.Buchstaber and T.Panov]

- K : simplicial complex
- $\mathscr{Z}_K$: moment angle complex of K

$$H^{**}(\mathscr{Z}_K;k)\cong\mathrm{Tor}_{k[v_1,\dots,v_m]}(k(K),k)$$

Thus we may have the following question.

When does $H^{**}(\mathscr{Z}_K;k)\cong H^{**}(\mathscr{Z}_L;k)$ as k-algebra imply $K\approx L$?

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