SUMMARY OF RESEARCH ACCOMPLISHMENTS

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1. Background

A *torus* is a classical compact abelian Lie group, i.e., a product of finitely many circles. *Toric topology* is the study of topological, algebraic, combinatorial, differential, geometric, and homotopy theoretic aspects of a particular class of torus actions. Since topological spaces with torus actions have nice symmetries and their quotients are very structured, toric topology have been studied over a century as an important sub-branch of equivariant topology. Recently, toric topology has become more important in various areas of mathematics such as algebraic geometry, symplectic geometry, convex geometry, combinatorics, differential topology, and homotopy theory.

The author, Suyoung Choi, has mostly worked on classification problems of toric objects both equivariantly and non-equivariantly and is interested in the relation between the topology of toric objects and its orbit space. He is also working on the topology of real analogues of toric objects. These studies, sometimes, are translated into the languages of convex polytope theory or graph theory which create new remarkable links between toric topology and combinatorics. Especially, his work makes a significant contribution to the rigidity problems for manifolds with nice torus actions and simplicial complexes as objects in toric topology. His mathematical tools are equivariant topology, algebraic topology, combinatorics, commutative and homological algebra, bundle theory, and convex geometry. He also uses a computer scientific approach to produce non-obvious examples in the areas.

This paper gives a summary of recent accomplishments by Choi with or without others in toric topology and the related areas.

2. Classification of toric objects

We recall the fact that if a compact torus T^k of dimension k acts on a smooth manifold of dimension n effectively with a fixed point, then we have $2k \leq n$. Hence, roughly speaking, the most symmetrical manifold is an even dimensional manifold admitting a half dimensional torus action. More precisely, a *torus manifold* introduced in [32] is a closed smooth orientable manifold of dimension $2n$ which admits an effective T^n-action with the non-empty fixed points set.

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One of the most important examples of torus manifold is a toric variety. A toric variety, which arose in the field of algebraic geometry, of dimension n is a normal algebraic variety with an action of an algebraic torus $(\mathbb{C}^*)^n$ having a dense orbit. A compact smooth toric variety is sometimes called a *toric manifold*. By regarding S^1 as the unit circle in $\mathbb{C}^*$, there is a natural action of $T^n = (S^1)^n \subset (\mathbb{C}^*)^n$ on a toric variety. Hence, a toric manifold is a torus manifold.

Instead of an algebraic torus action on an algebraic variety, one could think of a smooth torus (T^n or $(\mathbb{C}^*)^n$) action on a smooth manifold. Since the pioneering work of Davis and Januszkiewicz [26], a number of categories of manifolds which admit certain torus actions have been proposed as topological analogues of smooth toric varieties. A *quasitoric manifold* introduced in [26] is a closed smooth $2n$-manifold with an effective locally standard T^n-action whose orbit space can be identified with a simple polytope (which is dual to a simplicial polytope). A quasitoric manifold is surely a torus manifold. Moreover, every smooth projective toric manifold is a quasitoric manifold.

A *topological toric manifold* defined in [34] is a closed smooth $2n$-manifold M with an effective smooth $(\mathbb{C}^*)^n$-action such that there is an open and dense orbit and M is covered by finitely many invariant open subsets each of which is equivariantly diffeomorphic to a smooth representation space of $(\mathbb{C}^*)^n$. Every topological toric manifold is a torus manifold. Furthermore, every toric manifold and quasitoric manifold is a topological toric manifold by [34].

One interesting remark is that such toric objects are classified by pairs of a simplicial complex K (as the face complex of the orbit space) and a so-called characteristic map λ which contains the action information. Meanwhile, if K corresponds to a toric topological manifold, it should be a *fan-like* simplicial sphere, i.e, K is the underlying simplicial complex of a complete fan. Therefore, throughout this article, only fan-like simplicial spheres are considered.

To classify toric manifolds or topological toric manifolds, it seems natural to classify them with fixed K. Choi-Masuda-Suh [15] have studied toric manifolds and quasitoric manifolds over the join of the boundaries of simplices (dual to the product of simplices). Note that if B is a toric manifold and E is a Whitney sum of complex line bundles over B, then the projectivization $P(E)$ of E is again a toric manifold. Starting with B as a point and repeating this construction, we obtain a sequence of complex projective bundles which we call a *generalized Bott tower*, and especially call a *Bott tower* when each fiber is $\mathbb{C}P^1$. A Bott tower was introduced in [2]. The name of Bott tower was introduced and its study was initiated in [30]. In addition, the top manifold in the tower, called a (*generalized*) *Bott manifold*, is indeed a toric manifold and its orbit space can be identified with the join of the boundaries of simplices. Choi-Masuda-Suh [15] studied generalized Bott manifolds as quasitoric manifolds and found a necessary and sufficient condition for a quasitoric manifold over the join of the boundaries of simplices to be equivalent to a generalized Bott tower. (cf. [1], [27])

Recently, Choi and Park [20] have observed that the join of the boundaries of simplices is obtainable by (simplicial) wedge operations from a cross-polytope. Motivated this, they investigate the relationship between the topological toric manifolds over a simplicial complex K and those over the complex obtained by simplicial wedge operations from K. Their method is applicable to classify toric objects over K for the case when K is the join of boundaries of simplices or the case when $m-n$ is small, where m is the number of vertices of K. Furthermore, they classified toric objects over K when $m-n=3$ (which generalize a result of Batyrev [1] for toric manifolds), and gave a new and complete proof of projectivity of smooth toric varieties of Picard number 3 originally proved by Kleinschmidt and Sturmfels [36].

3. Topological classification problem: Cohomological rigidity problem

One of the most interesting problems in toric topology is the topological (or smooth) classification of toric manifolds. Interestingly, many recent researches provide evidences for toric manifolds to be classified by their cohomology rings. In general, the cohomology ring as an invariant is too weak to determine the topological type of a space. However, in the category of toric manifolds, we do not know any examples of two distinct toric manifolds having the same cohomology rings because of their tori-symmetries.

Hence, it raises the problem, called the *cohomological rigidity problem* for toric manifolds, which asks whether the topological types of toric manifolds are classified by their cohomology rings or not. The current research trend for the problem is summarized in [16].

Problem 1. *Are toric manifolds X and Y diffeomorphic if $H^*(X) \cong H^*(Y)$ as graded rings?*

By the classical theories on low dimensional manifolds such as [31, 40, 29], the answer to Problem 1 is affirmative for all toric manifolds up to 4 dimension since toric manifolds are simply connected. In high dimensional case, however, the problem is still open.

Indeed, in the last few years, the author obtained many substantial affirmative partial solutions to the problem which will be important intermediate steps for solving the problem.

Since the class of toric manifolds is too large to handle it is reasonable to restrict our attention to a smaller but an interesting subclass of manifolds. Thus, we may restrict our focus on (generalized) Bott manifolds or manifolds with small Betti numbers. We have some affirmative partial solutions to the problem for (generalized) Bott manifolds in [13, 17, 24, 33, 39]. Moreover, surprisingly, many results [13, 25, 33] show that all cohomology ring isomorphisms between the cohomology rings of two Bott manifolds under some conditions are induced by diffeomorphisms. Hence, for Bott manifolds, one can ask the so-called *strong cohomological rigidity problem* for Bott manifolds as the following.

Problem 2. *For (generalized) Bott manifolds X and Y, is any ring isomorphism $\varphi\colon H^*(X) \to H^*(Y)$ induced by some diffeomorphism $f\colon Y \to X$, i.e., $f^* = \varphi$?*

We remark that the answer to Problem 2 is negative for general toric manifolds (cf. [28]). The following are the summary of the result of Choi(and his collaborators), which provide partial affirmative solutions to Problems 1 and 2 for the following cases:

(1) $X = \prod_{i=1}^{m} \mathbb{C}P^{n_i}$ (rigidity: [17]), strong rigidity: [25]).
(2) X is a 3-stage Bott manifold (rigidity: [17], strong rigidity: [8]).
(3) X is a 4-stage Bott manifold (rigidity: [8]).
(4) X is a 2-stage generalized Bott manifolds (rigidity: [17], strong rigidity : [22]).
(5) X is a one-twisted Bott manifold (rigidity: [24]).
(6) X is a $\mathbb{Q}$-trivial Bott manifold (strong rigidity: [13]).
(7) X is a projective bundle over toric surface (rigidity: [21]).

All the cases above, Y can be an arbitrary toric manifold.

One can ask the cohomological rigidity problem for quasitoric manifolds (or topological toric manifolds) with homeomorphisms. In this case as well, no counter example has been known while some affirmative partial solutions are provided by Choi and his colleagues.

(8) X is a quasitoric manifold (or a topological toric manifold) with second Betti number 2 (rigidity: [23], strong rigidity: [22]).

One can also consider the rigidity problem for torus manifolds. Choi and Kuroki [10] have shown that torus manifolds with extended action of codimension one are classified topologically by their cohomology rings and characteristic classes. This provides also a counterexample to the rigidity problem for torus manifolds.

4. Toric rigidity of simplicial spheres

Let M be a toric object corresponding to a fan-like simplicial sphere K. We remark that the combinatorial structure of K is determined by the equivariant topology of M. In fact, by [3], it is known that the Stanley-Reisner ring of K, which is isomorphic to the equivariant cohomology ring of M, completely determines the combinatorial structure K. Moreover, since there is a projection from the equivariant cohomology rings to ordinary cohomology rings, the ordinary topology of M also contains some combinatorial information of K. For instance, the number of faces of K is completely determined by the Betti numbers of M (see [26]). However, in general, the topology of M does not contain sufficient information to determine K. Nevertheless, in many cases, K is determined by the ordinary cohomology ring $H^*(M)$ of M. For instance, Masuda and Panov [39] have shown that if $H^*(M)$ is isomorphic to the cohomology ring of some Bott manifold, then K is a cross-polytope. Motivated by this, the notion of cohomological rigidity of simplicial spheres arose.

A fan-like simplicial sphere K is (toric) *cohomologically rigid* (say, C-rigid) if there exists a topological toric manifold M over K, and whenever there exists a topological toric manifold N over L with a graded ring isomorphism $H^*(M;\mathbb{Z}) \cong H^*(N;\mathbb{Z})$, K is combinatorially isomorphic to L. Although $H^*(M)$ contains some information of K, not every simplicial sphere has this property, but some important complexes such as simplices or cross-polytopes are known to be cohomologically rigid.

Choi-Panov-Suh [18] investigated the cohomological rigidity of simplicial spheres. The main idea is that the ring isomorphism between the cohomology rings implies the algebra isomorphism between the Tor-algebra of K and L. Hence, a cohomological rigidity is related to the bigraded Betti numbers of its Stanley-Reisner ring, another important invariant coming from combinatorial commutative algebra.

The Tor-algebra of K is isomorphic to (bigraded) cohomology ring of the moment angle complex of K (see [5]). Hence, the result of [18] motivated another rigidity problem for the moment angle complexes (cf. [4]).

The main method in [18] also produces an interesting problem in polytope theory of combinatorics, which asks that which polytope can be determined by its Tor-algebra or its Betti numbers. Since the Betti numbers of Tor-algebra of K can be computed in purely combinatorial ways (cf. [41]), the following rigidity for simple polytopes can be understood as purely combinatorial properties of K.

- A simplicial sphere K is (toric) *algebraically rigid* (say, B-rigid) if L is isomorphic to K whenever their Tor-algebras are isomorphic as algebras.
- A simplicial sphere K is (toric) *combinatorially rigid* (say, A-rigid) if L is isomorphic to K whenever the Betti numbers of Tor-algebras are the same.

One can observe that

$$K \text{ is A-rigid} \Rightarrow K \text{ is B-rigid} \Rightarrow K \text{ is C-rigid},$$

where the last implication is provided if K supports a topological toric manifold.

Using this fact, Choi-Panov-Suh [18] found many examples of simplicial spheres which are A-rigid and therefore are cohomologically rigid.

These examples include the join of boundaries of simplices. Choi and Kim [12] investigated A-rigidity of 3-dimensional reducible polytopes.

Recently, Choi [9] himself found an example of simplicial sphere which is B-rigid but not A-rigid, and also found an example of simplicial sphere which is C-rigid but not B-rigid.

5. Real toric varieties and their topological analogues

Let M be a toric variety of complex dimension n. Then there is a canonical involution, called the *conjugation* of M. The set of its fixed points, denoted by $M_{\mathbb{R}}$, is a real subvariety of dimension n, called a *real toric variety*. When M is a toric manifold, then $M_{\mathbb{R}}$ is a submanifold of dimension n and called a *real toric manifold*.

A *real Bott manifold* is a real toric manifold obtained from a sequence of iterated $\mathbb{R}P^1$ bundles such that each fibration is the projective bundle of the Whitney sum of two real line bundles.

Choi [6] found a bijection between the set of real Bott manifolds of dimension n in the sense of [26] and the set of acyclic digraphs with n labeled nodes. Using this, he established the formula to compute the number of real Bott manifolds up to equivariant homeomorphism classes. Furthermore, he [7] also computed the number of orientable real Bott manifolds.

Choi-Masuda-Oum [14] completely characterized real Bott manifolds up to diffeomorphism in terms of three simple matrix operations on $(0, 1)$ matrices permutable to upper triangular form. This argument also proves that any graded ring isomorphism between the cohomology rings of real Bott manifolds with $\mathbb{Z}_2$ coefficients is induced by an affine diffeomorphism between the real Bott manifolds (cf. [35]). Their characterization can be visualized combinatorially in terms of graph operations on directed acyclic graphs based on the observation in [6]. Using this combinatorial viewpoint, they proved that the decomposition of a real Bott manifold into a product of indecomposable real Bott manifolds is unique up to permutations of the indecomposable factors, and also produced some numerical invariants of real Bott manifolds.

One interesting class of real toric varieties is the ones corresponding to so called graph-associahedra. Choi and Park [19] compute their the (rational) Betti numbers and Euler characteristic numbers. Interestingly, they can be calculated by a purely combinatorial method (in terms of graphs) which produces new invariants of graphs.

The concept of real toric varieties can be generalized to topological toric case. We say that a closed smooth manifold M of dimension n with an effective smooth action of $(\mathbb{R}^*)^n$ having an open dense orbit is a *real topological toric manifold* if it is covered by finitely many invariant open subsets each of which is equivariantly deffeomorphic to a direct sum of real one-dimensional smooth representation spaces of $(\mathbb{R}^*)^n$. In [20], Choi and Park classify and count real topological toric manifolds over K with $m - n = 3$. Furthermore, they proved that, when $m - n \leq 3$, any real topological toric manifold is realizable as fixed points of the conjugation of a topological toric manifold.

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