SUMMARY OF RESEARCH ACHIEVEMENTS

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A torus is a classical compact abelian Lie group, i.e., a product of finitely many circles. *Toric topology* is the study of topological, algebraic, combinatorial, differential, geometric, and homotopy theoretic aspects of a particular class of torus actions. Since topological spaces with torus actions have nice symmetries and their quotients are very structured, they have been studied for over a century as an important subbranch of equivariant topology. Recently, the study of them has become more important in various areas of mathematics such as algebraic geometry, combinatorial and convex geometry, commutative and homological algebra, differential topology, and homotopy theory. The author, Suyoung Choi, works on toric topology with the viewpoint of algebraic topology, commutative and homological algebra, and combinatorics. These studies, sometimes, can be translated into the language of convex polytope theory or graph theory. This article summarizes recent achievements of Choi, partially joint with other colleagues and collaborators, in toric topology and related areas.

Toric and quasitoric manifolds over the product of simplices. A *toric manifold* is a non-singular compact complex algebraic variety with an algebraic torus action having a dense orbit. We consider a projective toric manifold and regard the compact torus T as the standard compact subgroup in the algebraic torus. Then T also acts on a projective toric manifold and there is a moment map whose image is a simple convex polytope. (In symplectic geometry, this polytope is called a *Delzant polytope.*) Moreover, one can see that the T-action is *locally standard*, that is, locally modeled by the standard T-action on $\mathbb{C}^n$, where n is the complex dimension of the toric manifold. By taking these two characteristic properties as a starting point, Davis and Januszkiewicz [DJ91] introduced the notion of quasitoric manifold as a topological generalization of projective toric manifold in algebraic geometry. A *quasitoric manifold* is a smooth closed manifold of dimension $2n$ with a locally standard T^n-action whose orbit space can be identified with a simple polytope.

Choi-Masuda-Suh [CMS10a] have studied toric manifolds and quasitoric manifolds over a product of simplices. Note that if B is a toric manifold and E is a Whitney sum of complex line bundles over B, then the projectivization $P(E)$ of E is again a toric manifold. Starting with B as a point and repeating this construction, we obtain a sequence of complex projective bundles which we call a *generalized Bott tower*, and especially call a *Bott tower* when each fiber is $\mathbb{C}P^1$. A Bott tower was first introduced in[BS58]. The name of Bott tower was introduced and its study was initiated in [GK94]. In addition, the top manifold in the tower, called a (*generalized*) *Bott manifold*, is indeed a toric manifold and its orbit space can be identified with a

Date: May 12, 2010.

product of simplices. Choi-Masuda-Suh [CMS10a] found a necessary and sufficient condition for a quasitoric manifold over a product of simplices to be equivalent to a generalized Bott tower. (cf. [Dob01])

Topological classification problem: Cohomological rigidity problem. One of the most interesting problems in toric topology is the topological classification of toric manifolds. Interestingly, many recent researches provide evidences for toric manifolds to be classified by their cohomology rings. In general, the cohomology ring as an invariant is too weak to determine the topological type of a space. However, in the category of toric manifolds, we do not know any examples of two distinct toric manifolds having the same cohomology rings because of their tori-symmetries.

Hence, it raises the problem called the *cohomological rigidity problem* for toric manifolds, which asks whether the topological types of toric manifolds are classified by their cohomology rings or not (see [MS08]).

Problem 1. *Are toric manifolds X and Y diffeomorphic (or homeomorphic) if $H^*(X) \cong H^*(Y)$ as graded rings?*

By the classical theories on low dimensional manifolds such as [Hir51, OR70, Fre82], the answer to Problem 1 is affirmative for all toric manifolds up to 4 dimension since toric manifolds are simply connected. In high dimensional case, however, the problem is still open. Since the class of toric manifolds is too large to handle it is reasonable to restrict our attention to a smaller but an interesting subclass of manifolds. Thus, we may restrict our focus on (generalized) Bott manifolds.

We have some affirmative partial solutions to the problem for (generalized) Bott manifolds in [CM09, CMS10b, CS09a, Ish10, MP08]. Moreover, surprisingly, many results [CM09, CS09b, Ish10] show that all cohomology ring isomorphisms between the cohomology rings of two Bott manifolds under some conditions are induced by diffeomorphisms. Hence, for Bott manifolds, one can ask the so-called *strong cohomological rigidity problem* for Bott manifolds as the following.

Problem 2. *For (generalized) Bott manifolds X and Y, is any ring isomorphism $\varphi\colon H^*(X) \to H^*(Y)$ induced by some diffeomorphism $f\colon Y \to X$, i.e., $f^* = \varphi$?*

We remark that the answer to Problem 2 is negative for general toric manifolds (cf. [FM88]). The following are the summary of the result of Choi and his colleagues, which provide partial affirmative solutions to Problems 1 and 2.

(1) $X = \prod_{i=1}^{m} \mathbb{C}P^{n_i}$ and Y is an arbitrary toric manifold (rigidity: [CMS10b]), strong rigidity: [CS09b]).
(2) X is a 3-stage Bott manifold and Y is an arbitrary toric manifold (rigidity: [CMS10b], strong rigidity: in preparation).
(3) X is a 4-stage Bott manifolds and Y is an arbitrary toric manifold (rigidity: in preparation).
(4) Both X and Y are 2-stage generalized Bott manifolds (rigidity: [CMS10b]).
(5) X is a one-twisted Bott manifold and Y is an arbitrary toric manifold (rigidity: [CS09a]).

(6) X is a $\mathbb{Q}$-trivial Bott manifold and Y is an arbitrary toric manifold (strong rigidity: [CM09]).

On the other hand, any diffeomorphism between closed manifolds preserves Pontrjagin classes. Moreover, it is proved by Novikov [Nov65] that a homeomorphism also preserves rational Pontrjagin classes. Since the cohomology group of a Bott manifold has no torsion, any homeomorphism between toric manifolds also preserves their integral Pontrjagin classes. In fact, Choi [Cho10a] proved that any ring isomorphism between the cohomology rings of two Bott manifolds preserves the Pontrjagin classes. Hence, this result also provides an affirmative evidence to the strong cohomological rigidity problem for Bott manifolds. Moreover, by [Sul77], the diffeomorphism type of a simply connected Kähler manifold ($\dim_{\mathbb{C}} > 2$) is finitely determined by its integral cohomology ring and the Pontrjagin classes. Since a Bott manifold is projective, and hence, Kähler, the result in [Cho10a] implies that there are finitely many Bott manifolds having the same cohomology rings (cf. [McD10]).

One can ask the cohomological rigidity problem for quasitoric manifolds. Choi-Park-Suh [CPS10b] gave a partial answer to it.

(7) Both X and Y are quasitoric manifolds with second Betti number 2.

As an ultimate generalization of both toric and quasitoric manifolds, Hattori and Masuda [HM03] introduced a *torus manifold* (or *unitary toric manifold* in the earlier terminology [Mas99]) which is an oriented, closed, smooth manifold of dimension $2n$ with an effective T^n-action having a non-empty fixed point set. Problem 1 for torus manifolds whose quotients are highly structured is listed in [MS08]. Choi and Kuroki [CK09] gave a negative answer to it.

Toric cohomological rigidity of simple polytopes. Davis and Januszkiewicz [DJ91] proved that the equivariant cohomology ring of quasitoric manifold is isomorphic to the face ring of its orbit space. Since there is a projection from the equivariant cohomology rings to ordinary cohomology rings, the cohomology ring also contains some information of the orbit space.

A simple polytope P is (toric) *cohomologically rigid* if its combinatorial structure is determined by the cohomology ring of a quasitoric manifold M over P, i.e., there exists a quasitoric manifold M over P, and whenever there exists a quasitoric manifold N over another polytope Q with $H^*(M) = H^*(N)$ there is a combinatorial equivalence $P \approx Q$. Although $H^*(M)$ contains some information of P, not every simple polytope has this property, but some important polytopes such as simplices or cubes are known to be cohomologically rigid.

Masuda and Suh [MS08, Problem 2] asked whether a product of simplices is cohomologically rigid or not. Choi-Panov-Suh [CPS10a] investigated the cohomological rigidity of polytopes and establish it for several new classes of polytopes including products of simplices. The main idea is that the ring isomorphism between the cohomology rings implies the algebra isomorphism between the Tor-algebra of P and Q. Hence, a cohomological rigidity is related to the bigraded Betti numbers of its Stanley-Reisner ring, another important invariant coming from combinatorial commutative algebra.

The Tor-algebra of a simple polytope P is known to be isomorphic to (bigraded) cohomology ring of the moment angle complex of P, see [BP02]. Hence, the result of [CPS10a] motivated another rigidity problem for the moment angle complexes (cf. [Buc08]).

The main method in [CPS10a] also arises an interesting problem in polytope theory of combinatorics, which asks that which polytopes can be determined by their bigraded Betti numbers. Since bigraded Betti numbers can be computed in purely combinatorial ways (cf. [Sta83]), the cohomological rigidity problem for simple polytopes can be understood as a pure combinatorial problem. Choi and Kim [CK10b] investigated this problem for 3-dimensional reducible polytopes, and they provided many examples which are cohomologically rigid. Independently, they [CK10a] established a useful method to compute bigraded Betti numbers for polytopes and computed them for stacked polytopes by using their new method.

Real Bott manifolds and acyclic digraphs. Now, we consider the real analogue of Bott manifolds. A manifold is called a *real Bott manifold* if there is a sequence of iterated $\mathbb{R}P^1$ bundles such that each fibration is the projective bundle of the Whitney sum of two real line bundles.

Choi [Cho08] found a bijection between the set of real Bott manifolds of dimension n in the sense of [DJ91] and the set of acyclic digraphs with n labeled nodes. Using this, he established the formula to compute the number of real Bott manifolds up to equivariant homeomorphism classes. Furthermore, he [Cho10b] also computed the number of orientable real Bott manifolds.

Choi-Masuda-Oum [CMO10] (cf. [Mas08, CO10]) completely characterized real Bott manifolds up to diffeomorphism in terms of three simple matrix operations on $(0, 1)$ matrices permutable to upper triangular form. This argument also proves that any graded ring isomorphism between the cohomology rings of real Bott manifolds with $\mathbb{Z}_2$ coefficients is induced by an affine diffeomorphism between the real Bott manifolds. (cf. [KM09]).

Their characterization can be visualized combinatorially in terms of graph operations on directed acyclic graphs based on the observation in [Cho08]. To our surprise, one of the operations corresponds to a known operation in graph theory called a *local complementation*. A local complementation on digraphs was first introduced by Bouchet [Bou87]. This operation also appears in many other areas such as the coding theory and quantum information theory. The result adds another application of this operation in the topology. At the same time, this observation leads us to new interpretation of real Bott manifolds.

Using this combinatorial viewpoint, Choi-Masuda-Oum proved that the decomposition of a real Bott manifold into a product of indecomposable real Bott manifolds is unique up to permutations of the indecomposable factors, and also porduced some numerical invariants of real Bott manifolds.

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